Deflection formulae for beams and properties for equivalent beam models for parallel chord trusses and vierendeel frames

1. General formula

\[ \Delta = \Delta_b + \Delta_s = \frac{C_b WL^3}{EI} + \frac{C_s WL}{K_s} \]

- \( \Delta \) : deflection
- \( \Delta_b \) : deflection due to bending deformation
- \( \Delta_s \) : deflection due to shear deformation
- \( C_b \) : properties for equivalent beam models
- \( C_s \) : properties for equivalent beam models
- \( W \) : total load
- \( L \) : span
- \( EI \) : bending stiffness parameter
- \( K_s \) : shear stiffness

Expressions for \( K_s \) see Table 1

Expressions for \( I \) see Table 1

2. Beam

Table 1: Expressions for \( I \) and \( K_s \)

<table>
<thead>
<tr>
<th>Structure</th>
<th>( I )</th>
<th>( K_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam</td>
<td>( I )</td>
<td>( A_s G ) - Table 2</td>
</tr>
<tr>
<td>Parallel chord truss</td>
<td>( I_g )</td>
<td>( K_{st} ) - Eq (A)</td>
</tr>
<tr>
<td>Vierendeel frame</td>
<td>( I_g )</td>
<td>( K_{sv} ) - Eq(B)</td>
</tr>
</tbody>
</table>

Table 2: Values for shear area \( A_s \)

<table>
<thead>
<tr>
<th>Section</th>
<th>( A_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle ( b \times d )</td>
<td>( \frac{5}{6} bd )</td>
</tr>
<tr>
<td>I section bent about major axis</td>
<td>Area of web</td>
</tr>
<tr>
<td>I section bent about minor axis</td>
<td>( \frac{5}{6} ) Area of flanges</td>
</tr>
</tbody>
</table>

\[ I_g = \frac{A_s b^2}{2} \]

3. Parallel chord truss

\[ K_{st} = \frac{1}{fE_d A_d \sin^2 \theta \cos \theta} + \frac{1}{E_p A_p \cot \theta} \]

- \( f = 1.0 \) for singly braced truss
- \( f = 2.0 \) with compressive cross bracing
- \( f = 0.5 \) for K bracing

With tensile only cross bracing treat as singly braced
With compressive cross bracing ignore flexibility of posts
4. Vierendeel frame

\[ K_{sv} = \frac{24EI_c}{a^2[1 + 2\psi]} \]  

(B)

\[ \psi = \frac{l_c}{a} / \frac{l_p}{b} \]

Parameters for vierendeel frame

Table 3 Beam deflection coefficients

<table>
<thead>
<tr>
<th>Structure</th>
<th>Load</th>
<th>( C_b ) bending</th>
<th>( C_s ) shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilever</td>
<td>Point tip</td>
<td>1/3</td>
<td>1.0</td>
</tr>
<tr>
<td>E,I</td>
<td>UD</td>
<td>1/8</td>
<td>1/2</td>
</tr>
<tr>
<td>Simply supported</td>
<td>Point central</td>
<td>1/48</td>
<td>1/4</td>
</tr>
<tr>
<td>E,I</td>
<td>UD</td>
<td>5/384</td>
<td>1/8</td>
</tr>
</tbody>
</table>

5. Derivation of \( K_{st} \)

From the Bar Element Document, Equation (21) is:

\[ \Delta = \delta_1 + \delta_2 = \frac{WL_d}{(EA)_d \cos^2 \theta} + \frac{WL_p}{(EA)_p} \]

Governing differential equation for shear deformation:

\[ S = K_s \frac{dv}{dx} \]

i.e. \( K_s = \frac{S}{dv/dx} \)

\( v \) is the displacement in the y direction

From the diagram: \( dv/dx = \Delta/a \)

Substituting this and \( S = W \) into (21):

\[ \frac{dv}{dx} = \frac{SL_d}{a E_d A_d \sin^2 \theta} + \frac{Sb}{a E_p A_p} \]

Note that \( \sin^2 \theta \) is used because the \( \theta \) for the frame is (90 - \( \theta \)) for Equation (21)

Using \( a/L_d = \cos \theta \) and \( a/b = \cot \theta \):

\[ K_s = \frac{S}{dv/dx} = \frac{1}{E_d A_d \sin^2 \theta \cos \theta} + \frac{1}{E_p A_p \cot \theta} \]
6. Derivation of $K_{sv}$

Figure 1(a) shows a vierendeel frame with points of contraflexure at mid-length of all members. Such positions for the points of contraflexure is the fundamental assumption in developing the shear mode deformation of a vierendeel frame as an equivalent beam.

Also shown on Figure 1(a) is a section of the frame bounded by points of contraflexure. This is extracted to Figure 1(b) where the shear at the points of contraflexure $S/2$. $S$ is the total shear at the section and half is taken by each chord (assuming them to have the same $I$ value). The final trick is to work on a symmetrical half of this sub-frame as in Figure 1(c). The deflection under the $S/2$ load of the frame of Figure 1(c) is calculated (using the principle of virtual work) to be:

$$\Delta = \frac{S a^3}{24 E I_c} [1 + 2\psi] \quad \text{where} \quad \psi = \frac{l_p / a}{l_p / b}$$

hence $dv/dx = \Delta/a$ and $K_{sv} = S/(dv/dx)$

hence $K_{sv} = \frac{24E I_c}{a^3[1 + 2\psi]}$

Note $G = E/(2(1+\nu))$ where $\nu$ is Poisson’s Ratio

7. Constitutive relationships for bending and shear:

Bending

$$M = EI \frac{d^2v}{dx^2} \quad \text{i.e.} \quad \frac{d^2v}{dx^2} = \frac{M}{EI}$$

Shear

$$S = K_s \quad \text{i.e.} \quad \frac{dv}{dx} = \frac{S}{K_s}$$

where $K_s$ is the shear stiffness.

For a beam $K_s = A_s G$ where $A_s$ is the shear area and $G$ is the shear modulus.

Table 3 shows shapes of shear force and bending moment diagrams and corresponding displacement diagrams.

For bending, the basic relationship needs to be integrated twice to get the displacement. Therefore the function for the displaced shape is two orders higher than that for the bending moment e.g. from Table 3 with UD load, the bending moment is parabolic whereas the displacement is quartic (fourth order).
For shear, the basic relationship needs to be integrated once to get the displacement and therefore, for example, with a point load, the shear is constant and the displacement is linear.

Table 4 Diagram shapes

<table>
<thead>
<tr>
<th>Loading</th>
<th>Shear force</th>
<th>Displacement</th>
<th>Bending Mom</th>
<th>Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>Constant</td>
<td>linear</td>
<td>linear</td>
<td>cubic</td>
</tr>
<tr>
<td>UD</td>
<td>linear</td>
<td>parabolic</td>
<td>parabolic</td>
<td>quartic</td>
</tr>
</tbody>
</table>

I MacLeod 16.07.13